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The transfer of heat in turbulent boundary layers with injection or suction: universal laws and Stanton number equations

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1. INTRODUCTION

THE INJECTION or suction of fluid into the turbulent boundary layer through a porous surface is an effective means of promoting the thermal protection of walls [1, 2]. The transpired fluid convects thermal energy away from the wall, improving the ability of surfaces to withstand a high-temperature environment. Despite this classical application, transpired turbulent boundary layers have also been extensively studied in connection with a variety of other applications. For example, injection or suction of fluid can be effectively used to reduce the drag of flows around bodies [3–5]. Also, variations of the transpiration problem are found when the injected fluid is a chemical reagent or when either evaporation or sublimation occurs.

For low speed flows, a good account of the transpiration phenomenon is given in the literature by a number of analyses of the problem [3–14]. Studies of the velocity boundary layer have established solid expressions for the law of the wall and for the law of the wake [3, 5]. These expressions have been derived for a range of flow conditions, yielding a bilogarithmic expression for the skin friction. This skin friction equation is much less sensitive than the momentum–integral equation to small changes in the flow parameters and so gives much more reliable results [8]. Unfortunately, for the thermal turbulent boundary layer with transpiration, no equivalent equation has been derived for predictions of the friction temperature. Most of the work has been conducted at Stanford University [1, 2, 6, 7] to the study of universal correlations for the temperature profile and for the prediction of Stanton number. These studies, together with the analyses of refs. [9–11], show that close to the wall the temperature profile has, as predicted by simple theoretical mixing-length based approaches, a logarithmic behaviour. In this case, the similarity parameters and the constants in the expressions must be such that their dependence on the Prandtl number is adequately taken into account [11].

This paper derives two new expressions for the law of the wall and shows how they can be extended to the defect layer simply by using Coles' function. The resulting temperature defect expressions yield bilogarithmic expressions for Stanton number similar to the skin friction equation which differ markedly from the previously proposed expressions and are hence presented here for the first time. The Stanton number has normally been evaluated by applying a correction factor to the unblown Stanton number, St_0, according to an expression derived by Spalding [12] in the sixties. The present formulation advances this formulation since it incorporates all the advantages of the skin friction equation [8], besides dispensing the previous knowledge of St_0. The expression for Stanton, however, is sensitive to the flow hydrodynamics through u_τ as well as to the transfer of heat through t_τ [2]. This means that inaccuracies in the prediction of u_τ and t_τ are immediately propagated into St in a cumulative manner. To avoid this difficulty we then propose an alternative approach where St can be evaluated independently from the knowledge of u_τ.

The analysis is carried out for two-dimensional, steady flows over aerodynamically smooth surfaces with no external pressure gradients. Extension of the present results to more complex flows will be published shortly.

2. THE THERMAL TURBULENT BOUNDARY LAYER ON A POROUS SURFACE

To study the effects of the transpiration on the boundary layer we divide the flow region into distinct parts where certain dominant effects can be used to derive simplified equations. The formulation of the transpiration problem basically differs from the solid surface problem in the sense that the inertia effects near the wall can no longer be neglected [1, 5, 8]. Therefore, for the near-wall dominated part of the flow, the approximate energy equation becomes

$$\frac{\partial \dot{Q}''}{\partial y} = \rho c_p v_w \frac{\partial t}{\partial y}. \tag{1}$$

This equation, together with the velocity solution [6], the mixing-length hypothesis

$$\dot{Q}'' = \rho c_p k_m k_t y^2 \frac{\partial u}{\partial y}\frac{\partial t}{\partial y} \tag{2}$$

and the boundary condition

$$\dot{Q}''(0) = \dot{Q}''_w = \rho c_p u_r t_r \tag{3}$$

yields

$$\frac{k_m}{k_t}\frac{2}{v_w^+}\left\{\left(\frac{v_w^+ t^+ + 1}{v_w^+ t_b^+ + 1}\right)^{k_t/2k_m}\left[\frac{v_w^+}{2}\left(\frac{1}{k_m}\ln\frac{y_a^+}{y_b^+\,Pr}\right)+\sqrt{(1+v_w^+ u_a^+)}\right] - \sqrt{(1+v_w^+ u_a^+)}\right\} = \frac{1}{k_t}\ln\frac{y^+\,Pr}{y_b^+\,Pr} \tag{4}$$

where v_w^+ denotes V_w/u_τ and the pair (y_b^+, u_b^+) is a constant of integration which must be determined experimentally. Parameters k_m and k_t, characteristic of the turbulence modelling, must also be determined experimentally. The above equation is the so-called law of the wall; it is here cast for the first time in this form.

An analysis of the data of refs. [5, 6] shows that the value of t^+ where the laminar-conductive and the turbulent solutions meet is nearly independent of the injection rate. Calling this value t_b^+, it can be determined by patching the laminar and the fully turbulent layer solutions for unblown flows. This results in

$$t_b^+ = 10. \tag{5}$$

Parameter y_b^+, obtained through the laminar-conductive layer solution

$$t^+ = \frac{1}{v_w^+}\left(e^{(v_w^+ y^+ Pr)} - 1\right) \tag{6}$$

NOMENCLATURE

A, B, B'	parameter in laws of the wall
Cf	skin friction coefficient
c_p	specific heat at constant pressure
k_m	von Karman's constant in velocity profile
k_t, k_t'	von Karman's constants in temperature profiles
Pr	Prandtl number
$\dot{Q}''$	heat flux
Re	Reynolds number
St	Stanton number
T	temperature
t	dimensionless temperature, $(T-T_w)/(T_\infty-T_w)$
t_τ	friction temperature
t^+, t^*	inner layer similarity temperatures, $(T-T_w)/t_\tau$, $t/\sqrt{(St)}$
U, V	velocity components
u_τ	friction velocity
v_w	V_w/U_∞
u^+, v_w^+, v_w^*	U/u_τ, V_w/u_τ, $V_\infty/U_\infty\sqrt{(St)}$
w	Coles' function
y^+, y^*	inner layer similarity coordinates, yu_τ/ν, $yU_\infty\sqrt{(St)}/\nu$.

Greek symbols

α	thermal diffusivity
δ_m, δ_t	velocity boundary layer thickness, temperature boundary layer thickness
Δ	enthalpy thickness
θ	momentum thickness
ν	kinetic viscosity
Π_t, Π_t'	temperature wake profiles
ρ	density
τ	shear stress.

Subscripts

a, b	constants of integration
m	velocity
t	temperature
w	conditions at wall
∞	external flow conditions.

is given by

$$y_b^+ = \frac{\ln v_w^+ t_b^+ + 1}{v_w^+ Pr}. \quad (7)$$

Of course, in the limit as $v_w \to 0$, equation (4) reduces to the solid surface solution. Results provided by equation (4) are compared with the experimental data of ref. [6] in Fig. 1. As can be seen, the agreement is good for the blowing data. The discrepancies found for the suction data were expected since in the experiments suction was applied over a long stretch of porous surface and, under this condition, the turbulent fluctuations in the boundary layer were partially or even completely removed, resulting in a not very well established turbulent boundary layer. Please note the very low values of Re for the suction data. It is worth stressing here that the temperature data agree much better with the present theory (equation (4)) than the corresponding velocity data agree with the equation derived by Simpson [5] if classical values of k_m (0.41) and A (5.0) are used to correlate the data.

Equation (4) can more generally be written as

$$F_t(V_w, (T-T_w), u_\tau, t_\tau) = f_t(y^+, Pr). \quad (8)$$

An extension of this expression to the defect layer can be obtained if we follow Stevenson [4] and make

$$F_t(V_w, (T-T_w), u_\tau, t_\tau) - F_m(V_w, (T_\infty - T_w), u_\tau, t_\tau) = g_t\left(\frac{y}{\delta_t}\right). \quad (9)$$

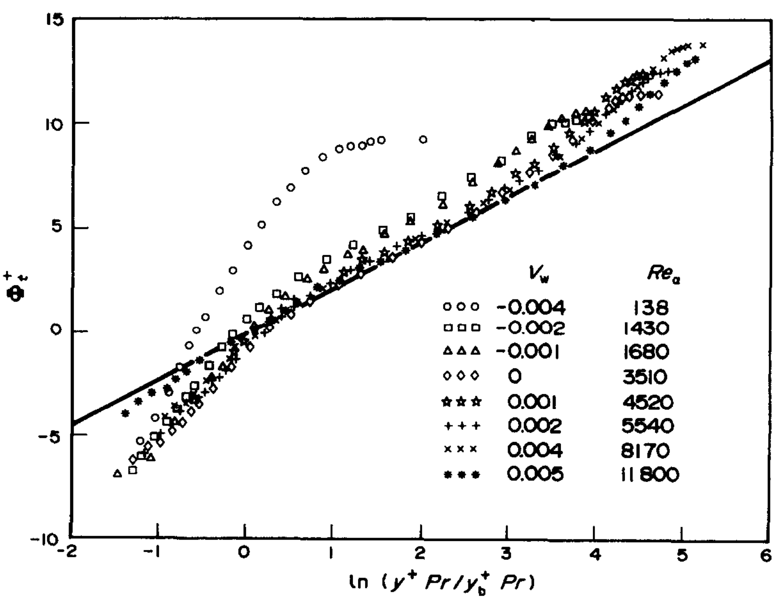

FIG. 1. Temperature law of the wall for transpired flow. Data from Whitten *et al.* [6].

Combination of equations (8) and (9) yields

$$\frac{2}{v_w^+}\left\{\left(\frac{v_w^+t^++1}{v_w^+t_b^++1}\right)^{k_t/2k_m}\left[\frac{v_w^+}{2}\left(\frac{1}{k_m}\ln\frac{y_a^+}{y_b^+\,Pr}\right)+\sqrt{(1+v_w^+u_a^+)}\right]-\sqrt{(1+v_w^+u_a^+)}\right\}=\frac{1}{k_m}\ln\frac{y^+\,Pr}{y_b^+\,Pr}+\frac{\Pi_t}{k_m}w\left(\frac{y}{\delta_t}\right). \quad (10)$$

Results obtained with the above formulation are shown in Fig. 2.

An equation for the prediction of the friction temperature is readily obtained if we substitute $(y,\ T) = (\delta_t,\ T_\infty)$ into equation (10) and obtain

$$\frac{T_w-T_\infty}{t_\tau}=\left\{\frac{\frac{2}{v_w^+}\left[\frac{1}{k_m}\ln\frac{y^+\,Pr}{y_b^+\,Pr}+\frac{\Pi_t}{k_t}w\left(\frac{y}{\delta_t}\right)\right]+\sqrt{(1+v_w^+u_u^+)}}{\frac{v_w^+}{2}\left(\frac{1}{k_m}\ln\frac{y_a^+}{y_b^+\,Pr}\right)+\sqrt{(1+v_w^+v_a^+)}}\right\}^{2k_m/k_t}\frac{(v_w^+t_b^++1)}{v_w^+}\frac{t_\tau}{T_w-T_\infty}\frac{1}{v_w^+}. \quad (11)$$

The problem can be given an alternative solution if we consider now the similarity variables defined by

$$t^*=\frac{T-T_w}{(T_\infty-T_w)\sqrt{(St)}},\ y^*=\frac{yU_\infty\sqrt{(St)}}{\nu}.$$

These similarity parameters were first suggested by Blackwell *et al.* [7]. They present the advantage of not including u_τ in their definitions so that St can be immediately evaluated, by the Stanton equation, from the main flow conditions. Then assuming that the velocity fluctuations, u'/U_∞, are proportional to the temperature fluctuations, $t'/\Delta T$, so that the mixing-length equation can be written as

$$\tau_{vt}=k_t'^2y^2\left(\frac{\partial t}{\partial y}\right)^2 \quad (12)$$

it follows that the law of the wall, equation (4), becomes

$$\frac{2}{v_w^*}[\sqrt{(1+t^*v_w^*)}-\sqrt{(1+t_b^*v_w^*)}]=\frac{1}{k_t'}\ln\frac{y^*\,Pr}{y_b^*\,Pr} \quad (13)$$

clearly a much simpler equation. Here we have $v_w^* = v_w/U_\infty\sqrt{(St)}$.

Extension of equation (13) to the defect region is again obtained by adding Coles' function to its right-hand side. Solutions with the alternative approach are shown in Figs. 3 and 4 together with the data of ref. [6].

For transpired flow, the Stanton equation can be written as

$$1=\sqrt{(St)}\left(\frac{1}{k_t'}\ln\frac{\delta_tU_\infty\sqrt{(St)}\,Pr}{\nu y_b^*Pr}+\frac{2\Pi_t'}{k_t'}\right)^2\frac{v_w^*}{4}+\sqrt{(St)}\times\left[\left(\frac{1}{k_t'}\ln\frac{\delta_tU_\infty\sqrt{(St)}\,Pr}{\nu y_b^*Pr}*+\frac{2\Pi_t'}{k_t'}\right)\sqrt{(1+t_b^*v_w^*)}+t_b^*\right]. \quad (14)$$

We also observed that the temperature wake profile varies with the flow conditions, as does the velocity wake profile. Indeed, we know that Π_m varies with Re_θ, its value departing from 0.0 and asymptotically approaching 0.55. Here we note that Π_t and Π_t' have the same qualitative behaviour of Π_m (see Figs. 5 and 6), varying asymptotically with Re_Δ (Δ = enthalpy thickness of the boundary layer). Both curves for Π_t and Π_t' suggest the asymptotic values to which these two parameters should tend. An insufficient number of experimental data, however, does not allow us to make a definitive assertion about these values.

Finally, predictions of Stanton number for several injection rates are tested against the data of Whitten *et al.* [6] in Table 1. This table also presents predictions obtained through the Spalding formulation [12], which yields

$$\frac{St}{St_0}=\frac{\ln\left(1+\frac{v_w}{St}\right)}{\frac{v_w}{St}}. \quad (15)$$

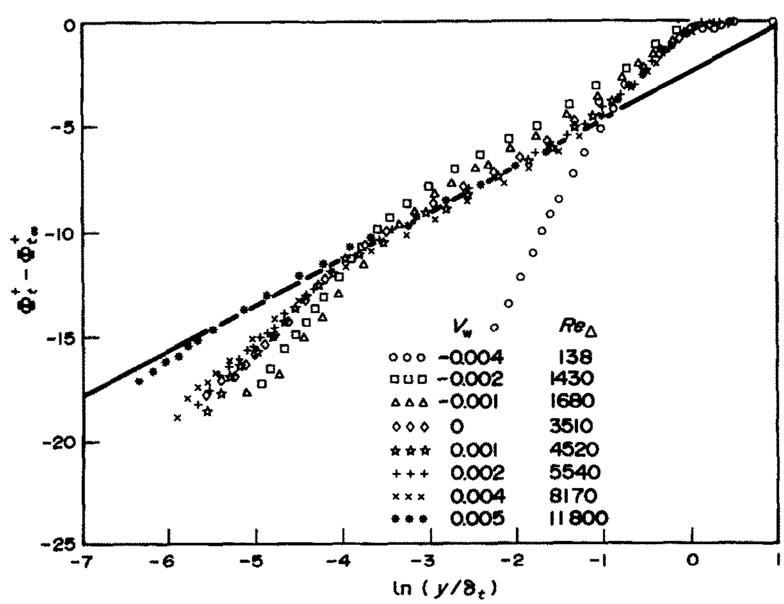

FIG. 2. Temperature law of the wake for transpired flow. Data from Whitten *et al.* [6].

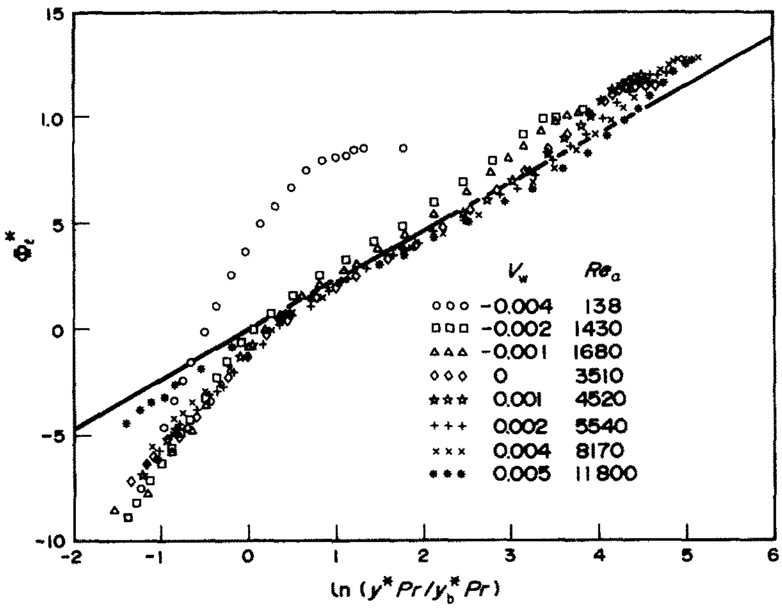

FIG. 3. Temperature law of the wall for transpired flow. Alternative approach. Data from Whitten *et al.* [6].

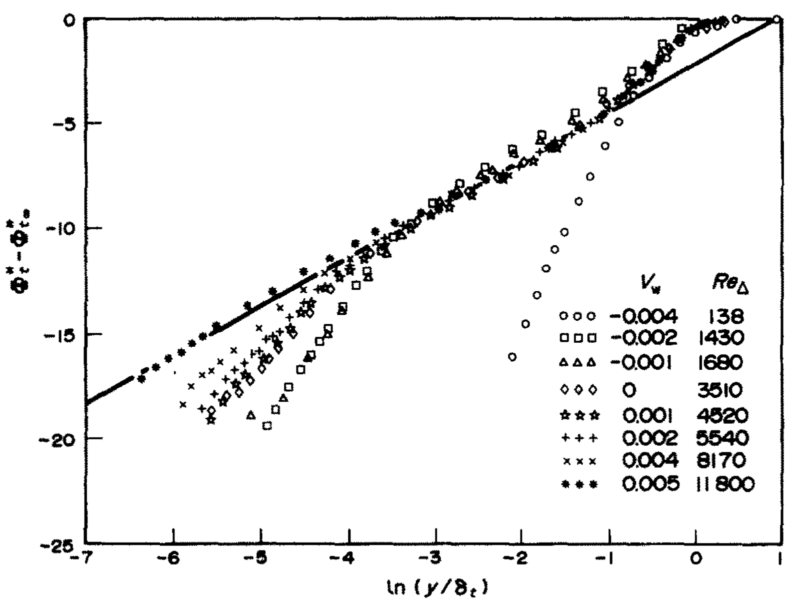

FIG. 4. Temperature law of the wake for transpired flow. Alternative approach. Data from Whitten *et al.* [6].

Of course application of equation (15) requires previous knowledge of St_0. Here we took values of St_0 from equation (IV-9) of ref. [6]. The present results are within 5% accuracy for the very high injection rates, improving the previous results which are 15% accurate. If experimental correlations for y_b^+ are used instead of equation (7) the results can be further improved.

3. CONCLUSION

In the present work, an analysis of the heat transfer problem for a transpired turbulent boundary layer has been performed. We have proposed new expressions for the law of the wall and for the law of the wake that give a good account of the temperature profile behaviour and which can be used to derive Stanton number equations. The overall agreement of the theory with the experimental data is good for both the temperature and the Stanton number predictions. This agreement is not better because we have opted for using analytic expressions for the determination of y_b^+ and y_b^* instead of considering any sort of experimental correlations. We have used expression (7) since this keeps our procedure self-contained. Of particular note is the good accuracy in the predictions of St for high injection rates.

Although comparisons are made just for air flow, we expect the present formulation to hold for any type of fluid. In fact, as suggested by the solid surface results of refs. [10, 11], we expect equation (4) to hold for other types of fluid

Table 1.

v_w	Re_Δ	St_{exp} [14]	St, equation (11)	St, equation (14)	St, equation (15)
-2.5×10^{-3}	9.6×10^2	3.46×10^{-3}	3.730×10^{-3}	3.683×10^{-3}	3.559×10^{-3}
−2.2	1.4×10^3	3.14	3.303	3.296	3.221
−1.1	1.7×10^3	2.63	2.791	2.744	2.762
0.0	3.5×10^3	1.95	1.955	1.903	
0.0	4.7×10^3	1.82	1.831	1.783	
0.9	5.7×10^3	1.43	1.472	1.398	1.455
1.0	4.5×10^3	1.52	1.565	1.475	1.546
1.8	7.1×10^3	1.12	1.167	1.072	1.129
1.9	5.5×10^3	1.23	1.238	1.124	1.179
3.7	1.1×10^4	0.66	0.658	0.560	0.596
3.9	8.2×10^3	0.69	0.720	0.604	0.640
4.8	1.2×10^4	0.50	0.476	0.380	0.399
5.0	9.4×10^3	0.53	0.518	0.412	0.441

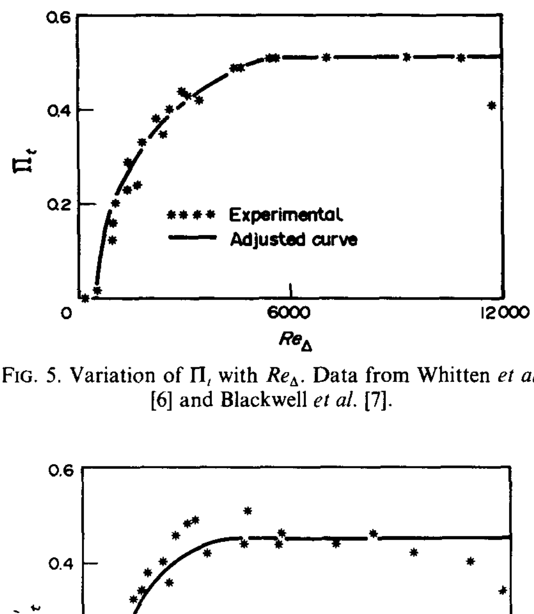

FIG. 5. Variation of Π_t with Re_Δ. Data from Whitten *et al.* [6] and Blackwell *et al.* [7].

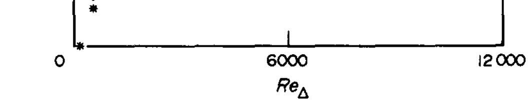

FIG. 6. Variation of Π_t' with Re_Δ. Data of Whitten *et al.* [6] and Blackwell *et al.* [7].

only if the level of the logarithmic curve is made to vary appropriately with *Pr*. Since far away from the wall the convective effects are dominant, we expect to obtain an extension of equation (4) to the defect layer by adding Coles' function to its right-hand side.

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